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EVOLUTION OF REPUTATION IN NETWORKS: A MEAN FIELD GAME APPROACH

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Abstract

This work models the competitive behaviour of individuals who maximize their own utility managing their network of connections with other individuals. Utility is taken as a synonym of reputation in this model. Each agent has to decide between two variables: the quality of connections and the number of connections. Hence, the reputation of an individual is a function of the number and the quality of connections within the network. On the other hand, individuals incur in a cost when they improve their network of contacts. The initial value of the quality and number of connections of each individual is distributed according to an initial (given) distribution. The competition occurs over continuous time and among a continuum of agents. A mean field game approach is adopted to solve the model, leading to an optimal trajectory for the number and quality of connections for each individual.

Key words: optimal stochastic control, mean field game, quality control, network analysis.

1 Introduction

In this work we describe the optimal behaviour of individuals who want to maximize their own utility, expressed in terms of reputation. It is important to consider that reputation is not an absolute value, but rather a relative one. In order to say whether an individual is of high or low reputation, it has to be compared with at least another individual. Thus, competition based on reputation is not based on a value *per se*, but on the position on a distribution of reputation. We shall see that this kind of competition based on the comparison between one's own position in the state space and the position (*i.e.* distribution) of the other individuals suits perfectly the structure and characteristics of Mean Field Game models. In our model individuals can increase their utility (*i.e.* reputation) through two instruments: the number and the quality of connections they have. Thus, the model tries to capture one of the fundamental aspect of reality, namely the importance of connections.

There are several practical situations that can illustrate this principle. First, consider a network where individuals are universities trying to improve their reputation. If an unknown university suddenly establishes partnerships with reputed ones such as Columbia, Oxford, Stanford, Princeton or Caltech, it is very likely that in a short period of time the perceived profile of that unknown university will increase sharply, becoming a top university. Here, this mechanism works as pure signalling: if the University was not very good to start with, it would never had the chance to partner with such top institutions.

Second, consider a low-skilled worker in a company whose network of relations includes the CFO, the Directors and other managers. It is likely that his/her salary will increase sharply in a short period of time, despite the fact that his/her own productivity does not change. However, his/her reputation (considered as a function of quality and number of connections) has increased very dramatically.

Third, consider an average firm that suddenly establishes strong connections with some of the best banks in the world. It is very likely that the reputation of that firm, which in this case can be expressed in terms of financial stability/performance or just the cost of debt, will raise significantly.

Although the three cases described above are quite extreme, the relevance of this study seems clear: the signalling mechanism underlying the networking strategy will work by considering the *quality* of the connections, and not only the *number* of connections. A university will always try to establish connections with more prestigious universities; an individual will always try to be well related to superiors in organizations; and firms will always try to cultivate their connections with natural financing sources. However, these attempts are costly. Our model tries to understand how individuals maximize their utility (their own reputation) given the incentives and the costs related to that mechanism.

The field of Mean Field Game Theory (MFGT) was first introduced by Jean-Michel Lasry and Pierre-Louis Lions (2006a, 2006b, 2007) and later developed by several scholars including Olivier Guéant (2009, 2013a, 2013b), whose applications of MFGT inspired the present work. MFGT was introduced as the limit case of stochastic differential games when the number of players goes to infinity. The innovative approach of the MFGT is based on the fact that for the first time an Hamilton-Jacobi-Bellman (HJB) equation and the forward Kolmogorov equation are doubly coupled. This means that individual behaviours enter in the Kolmogorov equation, which is not new, and, at the same time, the distribution of agents in the state space enters in the HJB equation, which is new. This turns out to be extremely useful for modelling economic problems with a large number of individuals. MFGT does not introduce a new paradigm, but rather constitutes a mathematical toolbox, enabling to analyse and understand strategic behaviour of individuals in a previously ignored environment.

The HJB equation expresses the maximization problem for each agent (or class of agents) in a continuous-time framework. The (forward) Kolmogorov equation enables to analyse and model how the optimal collective behaviour of the group of agents evolves over time. If these equations were singularly coupled, the collective behaviour of individuals would simply result from aggregating the individual optimal behaviour of each agent resulting from the HJB equation. However, something would be missing in such a setting. In this system, individuals would not take into considerations the actions of other agents when they chose their behaviour. Thus, the individual actions would be chosen independently and the Kolmogorov equation would just reflect their aggregate behaviour and how it would evolve over time. Under the MFGT framework the collective behaviour of agents in the state space enters as an ingredient in the HJB. Therefore, in the MFGT individuals are affected by the behaviour of the other agents. In particular, they are affected by the collective optimal behaviour (*i.e.* the distribution of agents in the state space) and its evolution over time.

The reason why is called Mean Field Game Theory is the following. First, the "Mean Field" part of the name comes from the fact that when individuals make decisions they are not concerned with the action of each individual, but with the distribution of the actions of all the other agents considered together. The name comes from Physics where the effect of all the other individuals on any given individual is approximated by a single averaged effect (a "field"), thus reducing a many-body problem to a one-body problem. Moreover, the "Game Theory" part of the name comes from the fact that Mean Field Games can be defined by increasing the number of players to infinity for a certain class of differential games. In particular, games in which players of the same kind can be interchanged turned out to be the one that best suited this passage to the limit. The main characteristics (hypothesis) of these kind of games is called invariance by permutation. Furthermore, Lasry and Lions (2006a, 2006b, 2007) showed that mean field games framework is effectively an approximation of a game with finite players when the number of players is large and that it is also possible to know the order of magnitude of the error when a mean field game model is adopted to solve a game with a finite number of players.

The model in this paper was inspired by the MFGT application presented by Guéant, Lasry and Lions (2011) describing the interaction between economic growth (measured as human capital accumulation) and the dynamics of inequality. In that model individuals have an incentive to improve their human capital in order to increase their wage and to avoid peer competition (*i.e.* competition from similar individuals). Although our work has some similarities with their work, it also has substantial differences. First, the different economic framework leads to a quite different utility function and costs function; second, our work adopts two state variables (number and quality of connections) instead of one, and also two controls instead of one leading to a more complex model to be solved.

This paper is organized as follows. Section 2 introduces the model, section 3 presents the resolution of the model, section 4 discusses the solution and section 5 concludes, pointing out limitations and suggesting ways to develop the model further.

2 The Model

We assume that individuals want to maximize their own reputation by controlling their number of connections h within a network, but also the quality q of their connections. We assume that the utility function should be proportional to $q_s^{\alpha} h_s^{\beta}$, where α and β are arbitrary positive constants, reflecting the (possibly different) weights of number and quality of connections.

Next, let $\bar{F}_1(s; q_s, l_s)$ be the survival function (or tail distribution) of q_s at time s, where l_s represents the lower bound of q_s (and of h_s). The survival function is given by one minus the cumulative distribution function of q_s at time s. As mentioned before for the case of reputation, the quality of connections strongly depends on the distribution of quality. Thus, the better the quality of connections that an individual has in comparison with the quality of other potential connections, the higher is its utility. Hence, utility should be proportional to $\frac{q_s^{\alpha} h_s^{\beta}}{\bar{F}(s; q_s, l_s)^{\varphi}}$.

Finally, utility should take into account peer competition. The higher the number of individuals with the same characteristics (same number and quality of connections), the higher is the competition between them, leading to lower signalling power. Thus, utility should decrease with peer competition. We follow Guéant, Lasry and Lions (2011) to introduce the impact of peer competition in the utility[1]. Let $m_1(s; q_s, l_s)$ denote the density function of q_s at time s, and $m_2(s; h_s, l_s)$ denote the density function of h_s at time s. The utility should then be proportional to the term $\frac{1}{m_1(s; q_s, l_s)^{\mu} m_2(s; h_s, l_s)^{\rho}}$, resulting in the final expression

$$G(s, q_s, h_s, l_s) = \frac{C}{m_1(s; q_s, l_s)^{\mu} m_2(s; h_s, l_s)^{\rho}} \frac{q_s^{\alpha} h_s^{\beta}}{\bar{F}_1(s; q_s, l_s)^{\varphi}},$$

where C is an arbitrary constant, and μ and ρ weight the relative importance of the state

[1]Guéant, Lasry and Lions (2011) uses the density function since in his model there was only one state variable. As in our case we do have two state variables, we might have used a joint density function. Instead we have assumed separate densities for technical convenience and modelling flexibility.

variables in peer competition.

Assume that for a given individual the initial values of quality and number of connections are $q_0 \geq l_0 = 1$ and $h_0 \geq l_0 = 1$. Then, the initial survival functions for q and h are assumed to be

$$\bar{F}_1(0; q_0, l_0) = \frac{1}{q_0^k}$$

and

$$\bar{F}_2(0; h_0, l_0) = \frac{1}{h_0^z},$$

where k and z are the parameters that determine the shape of the distributions, and the initial lower bound l_0 has been normalized to 1. These are survival functions generated by Pareto distributions. The use of the Pareto distribution is justified as follows. There is a large number of individuals with low and medium quality connections, a small number with good quality connections and a number even smaller with very good quality connections. Thus, the number of individuals with a given quality of connections is clearly a decreasing function of quality. Few universities have a connection with Harvard, while a larger number of universities have a connection with New York University and much larger with the University of Manchester.

Additionally we assume that the initial distribution of the number of connections is a Pareto distribution. The reason is that there are few universities that have several double degrees, while most of them have a small number. In particular, most of them have zero double degrees. The example can be extended to exchange programs using an appropriate scale. Moreover, there are few firms that have dozens of strong connections with banks, most of them have a small number. There are few workers that are very close friends with a lot of other workers or superiors in their workplace (*i.e.* in the same company or in the same sector they work), most of them have a small number of very close friends in their workplace.

From the $\bar{F}$ functions above, the probability density functions for q and h are given by

$$m_1(0; q_0, l_0) = k \frac{1}{q_0^{k+1}} \tag{1}$$

$$m_2(0;h_0,l_0) = z\frac{1}{h_0^{z+1}}. \tag{2}$$

On the other hand, individuals are constrained in their choice of q and h by a cost function. In order to understand the cost function it is necessary to first analyse the dynamics for q and h, assumed to follow the Stochastic Differential Equations (SDEs)

$$\begin{aligned} dq_s &= a(s,q_s)ds + \sigma q_s dW_s & (3) \\ dh_s &= b(s,h_s)ds + \sigma h_s dW_s, & (4) \end{aligned}$$

where $a(s,q_s)$ and $b(s,h_s)$ are the drift terms for q and h respectively, σq_s and σh_s are the diffusion terms for the processes above and q_0 and h_0 are the initial states. Additionally W_s denotes the usual Wiener process adapted with respect to a filtration $\mathscr{F}_t$. Following Guéant, Lasry and Lions (2011) we assume that the optimal drift functions simultaneously solving the HJB equation and the respective Kolmogorov equations are of the form:

$$\begin{aligned} a_{q_s} &\equiv a(s,q_s) = q_s\gamma & (5) \\ b_{h_s} &\equiv b(s,h_s) = h_s\gamma, & (6) \end{aligned}$$

meaning that both q and h follow a geometric Brownian motion. Using Itô's Lemma it follows that

$$\begin{aligned} q_s &= q_0 e^{(\gamma - \frac{\sigma^2}{2})s + \sigma W_s} = q_0 l_s & (7) \\ h_s &= h_0 e^{(\gamma - \frac{\sigma^2}{2})s + \sigma W_s} = h_0 l_s, & (8) \end{aligned}$$

where l_s is a third state variable, which is assumed to follow the SDE

$$dl_s = c(s,l_s)ds + \sigma l_s dW_s \tag{9}$$

with $l_0 = 1$ and

$$c_{l_s} \equiv c(s, l_s) = l_s \gamma. \tag{10}$$

Using Itô's Lemma we obtain

$$l_s = e^{(\gamma - \frac{\sigma^2}{2})s + \sigma W_s}.$$

In what follows we will verify whether this solution is correct or not. In particular, we are making the hypothesis that the number of connections (h) evolves over time with a constant common growth rate (γ). Moreover, we are making the same hypothesis for the quality of connections (q). However, for the quality of connections we are more interested in knowing how the realization of q is placed under the distribution $F(q)$ and less in its absolute value q.

Next, consider a cost function of the form

$$H(s, q_s, h_s, l_s) = R \frac{a(s, q_s)^{\varepsilon} b(s, h_s)^{\delta}}{\bar{F}_1(s; q_s, l_s)^{\omega} \bar{F}_2(s; h_s, l_s)^{\tau}}.$$

The logic of the structure of the cost function is the following. First, the presence of $b(s, h_s)$ and $a(s, q_s)$ means that individuals incur in a cost when they move in the state space. In particular, these costs depend on the level of q and h reached. According to our guess $a(s, q_s) = \gamma q_s$ and $b(s, h_s) = \gamma h_s$, increasing the quality and the number of connections becomes harder when the values of these variables are already high.

Second, the presence of $\bar{F}_1(s; q_s, l_s)$ and $\bar{F}_2(s; h_s, l_s)$ means that when an individual is closer to the frontier of q and h, it becomes harder to improve them. For instance, consider the difference in the difficulties faced by a university in establishing its second double degree today and one century ago, when double degrees did not exist. Thus, given the same value h, the costs depend on where that h is positioned in the distribution. When the number of connections is much higher than the number of connections others have, then it is difficult to increase it, most probably because the environment does not allow it. This applies also to the case of firms and their difficulty to make strong connections with a large number of banks, as well as to workers and the difficulty they face in being close to several top managers of the company in which they work.

Regarding the quality of the connections, being in the frontier makes it difficult to increase q, since there are few individuals with higher quality to be connected with. For instance, Stanford has a tremendous difficulty in increasing the quality of its connections because it is already connected to almost all top players: its q is already very high. This can be applied both to firms and their relationship with very good banks, and also to workers and their relations to very powerful friends. Moreover, $\bar{F}_1(s;q_s,l_s)$ and $\bar{F}_2(s;h_s,l_s)$ cover another important role in the costs' function, which reflects the exclusivity of the connections. If a university has connections of very high quality, then it will be more difficult to increase not only q but also h (without decreasing q), because increasing the number of connections can make prestigious universities to lose their status of exclusivity. In addition, when the number of connections h is very high, it will be more difficult to increase not only h but also q, because a university with too many connections has difficulty in increasing its quality of connections due to its lack of exclusivity. This is the reason why universities like Harvard will never have too many connections in comparison with most of the other universities.

Under the assumption in equations (5) and (6) Guéant, Lasry and Lions (2011) show that the probability density functions are given by[2]

$$\begin{aligned} m_1(s;q_s,l_s) &= k\frac{e^{k(\gamma-\frac{\sigma^2}{2})s+k\sigma W_s}}{q_s^{k+1}} = k\frac{l_s^k}{q_s^{k+1}} \qquad (11) \\ m_2(s;h_s,l_s) &= z\frac{e^{z(\gamma-\frac{\sigma^2}{2})s+z\sigma W_s}}{h_s^{z+1}} = z\frac{l_s^z}{h_s^{z+1}}, \qquad (12) \end{aligned}$$

given that $q_s \geq l_s$ and $h_s \geq l_s$. As a consequence, the survival function for the general stochastic case is shown to be given by

$$\bar{F}_1(s;q_s,l_s) = \frac{e^{k(\gamma-\frac{\sigma^2}{2})s+k\sigma W_s}}{q_s^k} = \frac{l_s^k}{q_s^k}$$

[2] An explicit proof of this result, which was not provided by Guéant, Lasry and Lions (2011), is presented in section 3.2 of this work.

$$\bar{F}_2(s;h_s,l_s) = \frac{e^{z(\gamma-\frac{\sigma^2}{2})s+z\sigma W_s}}{h_s^z} = \frac{l_s^z}{h_s^z},$$

given that $q_s \geq l_s$ and $h_s \geq l_s$. The equivalent expressions for the deterministic case can be simply obtained by making $\sigma = 0$.

We are now able to write down the Bellman function

$$J(t,q_t,h_t,l_t) = \max_{q_s,h_s} \mathbb{E}\left[\int_t^\infty [G(s,q_s,h_s,l_s) - H(s,q_s,h_s,l_s)]e^{-r(s-t)}ds|\mathscr{F}_t\right],$$

which can be rewritten as

$$J(t,q_t,h_t,l_t)$$
$$= \max_{q_s,h_s} \mathbb{E}\left\{\int_t^\infty \left[\frac{C}{m_1(s;q_s,l_s)^\mu m_2(s;h_s,l_s)^\rho}\frac{q_s^\alpha h_s^\beta}{\bar{F}_1(s;q_s,l_s)^\varphi} - R\frac{a(s,q_s)^\varepsilon b(s,h_s)^\delta}{\bar{F}_1(s;q_s,l_s)^\omega \bar{F}_2(s;h_s,l_s)^\tau}\right]e^{-r(s-t)}ds|\mathscr{F}_t\right\}.$$

Thus, substituting the values in the functions we get

$$J(t,q_t,h_t,l_t) = \max_{q_s,h_s} \mathbb{E}\left\{\int_t^\infty \left[\frac{C}{(k\frac{l_s^k}{q_s^{k+1}})^\mu (z\frac{l_s^z}{h_s^{z+1}})^\rho}\frac{q_s^\alpha h_s^\beta}{\left(\frac{l_s^k}{q_s^k}\right)^\varphi} - R\frac{a(s,q_s)^\varepsilon b(s,h_s)^\delta}{\left(\frac{l_s^k}{q_s^k}\right)^\omega \left(\frac{l_s^z}{h_s^z}\right)^\tau}\right]e^{-r(s-t)}ds|\mathscr{F}_t\right\}.$$

The solution for the optimal control problem is given by optimal values for the controls (which we guessed), the Bellman function and the probability density functions. In order to arrive to the solution, three Partial Differential Equations (PDEs) have to be satisfied: the HJB and the two Kolmogorov equations. We start the next section with the HJB equation.

3 Resolution of the Model

3.1 The Hamilton-Jacobi-Bellman equation

The Hamilton-Jacobi-Bellman is a fundamental element of *dynamic programming,* which is a powerful method for solving optimal control problems. This method, first developed by R. Bellman (1956), consists on a technique for making a sequence of interconnected decisions.

This technique can be applied to various maximization problems, including optimal control problems. The fundamental idea of this approach is the following. Take a class of optimal control problems with different initial time and states. Then, the HJB equation allows to establish the relationships among these problems and, in the case the HJB equation is solvable, it is possible to obtain an optimal feedback control by maximizing the Hamiltonian in the HJB equation.

In addition, it is important to mention that the solution of the HJB equation can be found either analytically or numerically and the optimal feedback control is a solution that holds for the whole class of optimal control problems (with different initial time and states), including the desired problem. Mathematically, the HJB equation is a *non-linear* partial differential equation of the first order (in the deterministic case) or of the second order (in the stochastic case). The Hamilton-Jacobi-Bellman equation for our problem is

$$\max_{a_q,b_h} \frac{C}{\left(k\frac{l^k}{q^{k+1}}\right)^{\mu}\left(z\frac{l^z}{h^{z+1}}\right)^{\rho}} \frac{q^{\alpha}h^{\beta}}{\left(\frac{l^k}{q^k}\right)^{\varphi}} - R\frac{a_q^{\varepsilon}b_h^{\delta}}{\left(\frac{l^k}{q^k}\right)^{\omega}\left(\frac{l^z}{h^z}\right)^{\tau}} - rJ + \partial_t J + a_q\partial_q J + b_h\partial_h J$$

$$+c_l\partial_l J + \frac{\sigma^2}{2}q^2\partial^2_{qq}J + \frac{\sigma^2}{2}h^2\partial^2_{hh}J + \frac{\sigma^2}{2}l^2\partial^2_{ll}J$$
$$+\sigma^2 qh\partial^2_{hq}J + \sigma^2 ql\partial^2_{ql}J + \sigma^2 hl\partial^2_{hl}J = 0.$$

Here q, h, and l are points of the trajectories (*i.e.* a realization of the stochastic processes) q_s, h_s, and l_s for each time s, with $s \in [t,\infty)$. Moreover, from equations (5), (6) and (10)

$$c_l \equiv c(t,l) = a(t,l) = b(t,l) = \gamma l,$$

given our guess $a(t,l) = b(t,l) = \gamma l$.

The optimal control functions are given by:

$$\arg\max_{a_q,b_h} a_q\partial_q J + b_h\partial_h J - R\frac{a_q^{\varepsilon}b_h^{\delta}}{\left(\frac{l^k}{q^k}\right)^{\omega}\left(\frac{l^z}{h^z}\right)^{\tau}}.$$

Hence, we have

$$-\varepsilon R\frac{(a_q^*)^{\varepsilon-1}(b_h)^{\delta}}{\left(\frac{l^k}{q^k}\right)^{\omega}\left(\frac{l^z}{h^z}\right)^{\tau}}+\partial_q J=0$$

$$-\delta R\frac{(a_q)^{\varepsilon}(b_h^*)^{\delta-1}}{\left(\frac{l^k}{q^k}\right)^{\omega}\left(\frac{l^z}{h^z}\right)^{\tau}}+\partial_h J=0.$$

Substituting the solution $a_q^*=\gamma q$ and $b_h^*=\gamma h$ we get for the derivatives

$$\begin{aligned}\partial_q J &= \varepsilon R\frac{(\gamma q)^{\varepsilon-1}(\gamma h)^{\delta}}{\left(\frac{l^k}{q^k}\right)^{\omega}\left(\frac{l^z}{h^z}\right)^{\tau}}\\ \partial_h J &= \delta R\frac{(\gamma q)^{\varepsilon}(\gamma h)^{\delta-1}}{\left(\frac{l^k}{q^k}\right)^{\omega}\left(\frac{l^z}{h^z}\right)^{\tau}}.\end{aligned}$$

Thus, taking the respective integrals we obtain

$$J = R\frac{\varepsilon}{\varepsilon+\omega k}\frac{q^{\varepsilon}h^{\delta}\gamma^{\varepsilon+\delta-1}}{\left(\frac{l^k}{q^k}\right)^{\omega}\left(\frac{l^z}{h^z}\right)^{\tau}} \tag{13}$$

$$J = R\frac{\delta}{\delta+\tau z}\frac{q^{\varepsilon}h^{\delta}\gamma^{\varepsilon+\delta-1}}{\left(\frac{l^k}{q^k}\right)^{\omega}\left(\frac{l^z}{h^z}\right)^{\tau}}. \tag{14}$$

In order to have a unique Bellman function we must impose $(13)=(14)$ implying that

$$\frac{\varepsilon}{\varepsilon+\omega k}=\frac{\delta}{\delta+\tau z},$$

which can be rewritten as

$$\varepsilon\tau z=\delta\omega k.$$

The Bellman function J can then be rewritten as

$$J=R\frac{\varepsilon}{\varepsilon+\omega k}\frac{q^{\varepsilon+\omega k}h^{\delta+\tau z}\gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}}.$$

It is important to stress the fact that the Hamiltonian has to be a concave function of a_q and b_h in order to ensure that we are in presence of a maximum.

Now, we are going to compute the value of the various elements of the HJB equation

$$
\begin{aligned}
\partial_t J &= 0 \\
c_l \partial_l J &= -R\gamma(\omega k + \tau z)\frac{\varepsilon}{\varepsilon + \omega k}\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}} \\
\frac{\sigma^2}{2} q^2 \partial^2_{qq} J &= \frac{\sigma^2}{2} R\varepsilon(\varepsilon + \omega k - 1)\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}} \\
\frac{\sigma^2}{2} h^2 \partial^2_{hh} J &= \frac{\sigma^2}{2} R\delta(\delta + \tau z - 1)\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}} \\
\frac{\sigma^2}{2} l^2 \partial^2_{ll} J &= \frac{\sigma^2}{2} R(\omega k + \tau z)(\omega k + \tau z - 1)\frac{\varepsilon}{\varepsilon + \omega k}\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}} \\
\sigma^2 q l \partial^2_{ql} J &= -\sigma^2 R\varepsilon(\omega k + \tau z)\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}} \\
\sigma^2 h l \partial^2_{hl} J &= -\sigma^2 R\delta(\omega k + \tau z)\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}} \\
\sigma^2 h q \partial^2_{hq} J &= \sigma^2 R\varepsilon(\delta + \tau z)\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}} \\
a_q \partial_q J &= R\varepsilon\gamma\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}} \\
b_h \partial_h J &= R\delta\gamma\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}}.
\end{aligned}
$$

Notice that $c_l \partial_l J$ is exogenous in the optimization, since individuals are atomized. Thus, they cannot affect $l(s)$, which is common for all individuals. Substituting these values into the HJB equation we obtain

$$
\begin{aligned}
&\frac{C}{k^\mu z^\rho}\frac{q^{\alpha+\varphi k+\mu(k+1)} h^{\beta+\rho(z+1)}}{l^{\varphi k+\mu k+\rho z}} - R\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta}}{l^{\omega k+\tau z}} - rR\frac{\varepsilon}{\varepsilon+\omega k}\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}} \\
&+R\varepsilon\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta}}{l^{\omega k+\tau z}} + R\delta\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta}}{l^{\omega k+\tau z}} - R(\omega k + \tau z)\frac{\varepsilon}{\varepsilon+\omega k}\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta}}{l^{\omega k+\tau z}} \\
&+\frac{\sigma^2}{2} R\varepsilon(\varepsilon+\omega k-1)\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}} + \frac{\sigma^2}{2} R\delta(\delta+\tau z-1)\frac{q^{\varepsilon+\omega k} h^{\delta+\tau z} \gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}}
\end{aligned}
$$

$$+\frac{\sigma^2}{2}R(\omega k+\tau z)(\omega k+\tau z-1)\frac{\varepsilon}{\varepsilon+\omega k}\frac{q^{\varepsilon+\omega k}h^{\delta+\tau z}\gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}}+\sigma^2 R\varepsilon(\delta+\tau z)\frac{q^{\varepsilon+\omega k}h^{\delta+\tau z}\gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}}$$

$$-\sigma^2 R\varepsilon(\omega k+\tau z)\frac{q^{\varepsilon+\omega k}h^{\delta+\tau z}\gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}}-\sigma^2 R\delta(\omega k+\tau z)\frac{q^{\varepsilon+\omega k}h^{\delta+\tau z}\gamma^{\varepsilon+\delta-1}}{l^{\omega k+\tau z}}=0.$$

Let us make the following additional assumptions on the parameters

$$\varphi k+\mu k+\rho z=\omega k+\tau z$$

$$\alpha+\varphi k+\mu(k+1)=\varepsilon+\omega k$$

$$\beta+\rho(z+1)=\delta+\tau z.$$

Then dividing both sides of the equation by $\frac{q^{\alpha+\varphi k+\mu(k+1)}h^{\beta+\rho(z+1)}}{Rl^{\varphi k+\mu k+\rho z}}$, and grouping the members of the equation according to the order of γ we get

$$\frac{C}{Rk^{\mu}z^{\rho}}+\gamma^{\varepsilon+\delta}\left[\varepsilon+\delta-1-(\omega k+\tau z)\frac{\varepsilon}{\varepsilon+\omega k}\right]$$

$$+\gamma^{\varepsilon+\delta-1}\left[\frac{\sigma^2}{2}\varepsilon(\varepsilon+\omega k-1)+\frac{\sigma^2}{2}\delta(\delta+\tau z-1)+\frac{\sigma^2}{2}(\omega k+\tau z)(\omega k+\tau z-1)\frac{\varepsilon}{\varepsilon+\omega k}\right.$$

$$\left.+\sigma^2\varepsilon(\delta+\tau z)-\sigma^2\varepsilon(\omega k+\tau z)-\sigma^2\delta(\omega k+\tau z)-r\right]=0. \qquad (15)$$

This equation has a unique positive solution for γ if $\varepsilon+\delta-1-(\omega k+\tau z)\frac{\varepsilon}{\varepsilon+\omega k}<0$ and r is high enough. The result will be discussed deeply in section 4.

3.2 The forward Kolmogorov equation

The forward Kolmogorov equation, also known as Fokker-Planck equation, can be defined as a partial differential equation of the first order (in the deterministic case) or of the second order (in the stochastic case) for the time evolution of the probability density function of a deterministic process or stochastic process. The classical formulation of the forward Kolmogorov equation for a density function $m_1(s;q_s)$ of the deterministic process $dq_s=a(t,q_s)ds$ (with

q_0 given) is

$$\partial_s m_1(s;q) + \partial_q \left[m_1(s;q)a_q\right] = 0,$$

with a given initial condition for $m_1(0;q_0)$, while for a density function $m_1(s;q_s,l_s)$ of the stochastic processes (3) and (9) (with q_0 and l_0 given) is

$$\partial_s m_1(s;q,l) + \partial_q \left[m_1(s;q,l)a_q\right] + \partial_l \left[m_1(s;q,l)c_l\right] = \frac{\sigma^2}{2}\partial^2_{qq}\left[q^2 m_1(s;q,l)\right]$$

$$+\frac{\sigma^2}{2}\partial^2_{ll}\left[l^2 m_1(s;q,l)\right] + \sigma^2\partial^2_{ql}\left[ql m_1(s;q,l)\right],$$

with a given initial condition for $m_1(0;q_0,l_0)$.

Let us first focus on the deterministic case. In our model, the initial condition is $m_1(0;q_0) = k\frac{1}{q_0^{k+1}}$ and using the expressions (5) and (11) we have

$$\partial_s\left(k\frac{e^{k\gamma s}}{q^{k+1}}\right) + \partial_q\left(k\gamma\frac{e^{k\gamma s}}{q^k}\right) = 0,$$

leading to

$$\gamma k^2\frac{e^{k\gamma s}}{q^{k+1}} - \gamma k^2\frac{e^{k\gamma s}}{q^{k+1}} = 0,$$

which is always trivially satisfied. Here and in what follows the *always satisfied* stands for *always satisfied for any* $s \in [t,\infty)$.

In the stochastic case, we have the initial condition (1) and we use the expressions (5), (10) and (11) to write

$$\partial_s\left(k\frac{l^k}{q^{k+1}}\right) + \partial_q\left(k\gamma\frac{l^k}{q^k}\right) + \partial_l\left(k\gamma\frac{l^{k+1}}{q^{k+1}}\right) = \frac{\sigma^2}{2}\partial^2_{qq}\left(k\frac{l^k}{q^{k-1}}\right) + \frac{\sigma^2}{2}\partial^2_{ll}\left(k\frac{l^{k+2}}{q^{k+1}}\right) + \sigma^2\partial^2_{ql}\left(k\frac{l^{k+1}}{q^k}\right).$$

The first term is zero since $k\frac{l^k}{q^{k+1}}$ does not depend on s. Using equation (7), all the other derivatives are zero and the Kolmogorov equation is trivially satisfied.

Moreover, the other condition to be satisfied is the following

$$m_1 > 0, \int m_1 dq = 1 \text{ for all } s \in [t, \infty).$$

For the deterministic case of our model we have that

$$m_1 = k\frac{e^{k\gamma s}}{q^{k+1}} > 0$$

and

$$\int m_1 dq = \int_{e^{\gamma s}}^{\infty} k\frac{e^{k\gamma s}}{q^{k+1}} dq = 0 - \left(-e^{k\gamma s}\frac{1}{e^{k\gamma s}}\right) = 1$$

which is always satisfied. For the stochastic case we have:

$$m_1 = k\frac{l^k}{q^{k+1}} > 0$$

and

$$\int m_1 dq = \int_{l}^{\infty} k\frac{l^k}{q^{k+1}} dq = 0 - \left(-l^k\frac{1}{l^k}\right) = 1$$

and both conditions are therefore always satisfied. The same procedure applies to $m_2(s; h_s, l_s)$ by making use of the initial condition (2) and the expressions (6) and (12).

3.3 The transversality condition

Now, we need to verify that the above solution satisfies the transversality condition, which is a boundary condition of the problem. The transversality condition for our problem consists in finding the range of values for the optimal growth rate γ such that the integrand is integrable (*i.e.* is bounded by a function with finite expected value) as $s \to \infty$. Thus, we have:

$$L = \left[\frac{C}{(k\frac{l_s^k}{q_s^{k+1}})^{\mu}(z\frac{l_s^z}{h_s^{z+1}})^{\rho}} \frac{q_s^{\alpha} h_s^{\beta}}{\left(\frac{l_s^k}{q_s^k}\right)^{\varphi}} - R\frac{(\gamma q_s)^{\varepsilon}(\gamma h_s)^{\delta}}{\left(\frac{l_s^k}{q_s^k}\right)^{\omega}\left(\frac{l_s^z}{h_s^z}\right)^{\tau}}\right] e^{-r(s-t)},$$

which can be rewritten as

$$L = \left[\frac{C}{Rk^{\mu} z^{\rho}} l_s^{\alpha+\beta+\mu+\rho} q_0^{\alpha+k\varphi+k\mu} h_0^{\delta+z\rho} - R\gamma^{\varepsilon+\delta} l_s^{\varepsilon+\delta} q_0^{\varepsilon+k\omega} h_0^{\delta+z\tau} \right] e^{-r(s-t)}$$

with $A = \frac{C}{Rk^{\mu} z^{\rho}} q_0^{\alpha+k\varphi+k\mu} h_0^{\delta+z\rho}$ and $B = R\gamma^{\varepsilon+\delta} q_0^{\varepsilon+k\omega} h_0^{\delta+z\tau}$. Now, define $M = \max(\varepsilon+\delta, \alpha+\beta+\mu+\rho)$. Then the above expression is limited by

$$L < l_s^M \left[A l_s^{\frac{\alpha+\beta+\mu+\rho}{M}} - B l_s^{\frac{\varepsilon+\delta}{M}} \right] e^{-r(s-t)}$$

Thus, the limiting behaviour of L is bounded by

$$\lim_{s\to\infty} L < K l_s^{M+1} e^{-r(s-t)}.$$

By Itô's Lemma we know that

$$\mathbb{E}[l_s^{M+1}] = e^{(M+1)(\gamma - \frac{\sigma^2}{2})s + \frac{1}{2}(M+1)^2\sigma^2 s},$$

and therefore, the transversality condition holds if

$$(M+1)(\gamma - \frac{\sigma^2}{2}) + \frac{1}{2}(M+1)^2\sigma^2 - r < 0$$

that is if

$$r > (M+1)(\gamma - \frac{\sigma^2}{2}) + \frac{1}{2}(M+1)^2\sigma^2$$

Thus, the parameter r has to be high enough in order to satisfy the Transversality condition. This is in accordance with the condition on r for the HJB equation to have a positive, unique optimal growth rate γ.

4 Results

4.1 Presentation of the Results

The results obtained for the stochastic optimal control problem are summarized in this section.

The optimal state trajectories for the quality q and the number h of connections grow exponentially as a Geometric Brownian Motion with a drift γ and diffusion coefficient σ

$$\begin{aligned} q_s &= q_0 l_s = q_0 e^{(\gamma - \frac{\sigma^2}{2})s + \sigma W_s} \\ h_s &= h_0 l_s = h_0 e^{(\gamma - \frac{\sigma^2}{2})s + \sigma W_s} \end{aligned}$$

where, under certain assumptions, γ is positive and unique, and is given by the HJB equation.

Moreover, the Bellman function is given by

$$J(q,h,l) = R \frac{\varepsilon}{\varepsilon + \omega k} \frac{q^{\varepsilon + \omega k} h^{\delta + \tau z} \gamma^{\varepsilon + \delta - 1}}{l^{\omega k + \tau z}}$$

and the optimal probability density function is

$$m_1(s; q_s, l_s) = k \frac{l_s^k}{q_s^{k+1}} = k \frac{e^{k(\gamma - \frac{\sigma^2}{2})s + k\sigma W_s}}{q_s^{k+1}}$$

for the quality of connections and

$$m_2(s; h_s, l_s) = z \frac{l_s^z}{h_s^{z+1}} = z \frac{e^{z(\gamma - \frac{\sigma^2}{2})s + z\sigma W_s}}{h_s^{z+1}}$$

for the number of connections.

4.2 Discussion of the Assumptions of the Model

First of all, the guess we initially made on the optimal feedback control is correct and the optimal growth rate is positive and unique, under some assumptions on the parameters. Thus, it is important to have a look at all the assumptions made. They are the following.

The first assumption is that there must be a unique Bellman function for the problem. Since there are two ways of obtaining this function, either through expression (13) or through expression (14), the assumption that both expressions should coincide imposes the following restriction on the parameters

$$\varepsilon \tau z = \delta \omega k.$$

Also, in order to get a closed-form solution, we have assumed that the power of q is the same in all terms of the HJB equation, leading to a restriction on the parameters of the form

$$\varphi k + \mu k + \rho z = \omega k + \tau z.$$

Similarly, we assumed that the power of h is the same in all terms of the HJB equation, leading to

$$\beta + \rho(z+1) = \delta + \tau z$$

and also for all the powers of l leading to

$$\alpha + \varphi k + \mu(k+1) = \varepsilon + \omega k.$$

Further, the conditions that we have to impose on the HJB solution for the drift γ to be unique are that

$$\varepsilon + \delta - 1 - (\omega k + \tau z)\frac{\varepsilon}{\varepsilon + \omega k} < 0$$

and a sufficiently large value for r

$$r > \sigma^2 \varepsilon(\omega k + \tau z) + \sigma^2 \delta(\omega k + \tau z) - \sigma^2 \varepsilon(\delta + \tau z)$$

$$-\frac{\sigma^2}{2}\varepsilon(\varepsilon+\omega k-1)-\frac{\sigma^2}{2}\delta(\delta+\tau z-1)-\frac{\sigma^2}{2}(\omega k+\tau z)(\omega k+\tau z-1)\frac{\varepsilon}{\varepsilon+\omega k}.$$

From the transversality condition, the solution holds if r is sufficiently large,

$$r>(M+1)(\gamma-\frac{\sigma^2}{2})+\frac{1}{2}(M+1)^2\sigma^2.$$

Despite it seems that this high number of assumptions is a weak point of the model, in fact it is not. If we look deeply at the constraints we realize that the last two inequalities are satisfied for r high enough, and that the other inequality can be easily satisfied given the high number of parameters available. Looking at the equalities above, it is possible to see that we have 4 equations and 11 parameters. Therefore, the model has a very high number of degree of freedom, which makes it feasible for calibration in empirical studies.

4.3 Discussion of the Results

In this section we describe in words the results obtained above and discuss their fit to reality. In order to address this issue we start by focusing on the example of universities. One of the aspects of the solution is that the number of connections of an individual increases on average over time. In the last century and in particular in the last 50 years the number of connections between universities (exchange programs, double degrees and partnerships, among others) increased dramatically. This phenomenon happened for all different types of universities. Thus, the model is able to explain this fact.

The model also suggests that universities with a low number of initial connections (h_0 close to the lower bound) evolve to remain those with the lower number of connections. It is possible to see this empirically by looking at the evolution of h for very good universities in the last 30 years. Initially they had a small number of connections in comparison with other universities, due to their exclusivity. Their number of connections is clearly higher than 30 years ago, but MIT still has much less connections than the University of Granada in Spain. Therefore, the model is able to predict that a university like MIT had just a few partnerships in 1980 may be now actively connected with about 30 institutions in 12 countries, whereas

Bologna University had tens of partnerships in 1980 and it now has around 3000 as reported in their website.

Furthermore, even extreme cases of top reputation like Harvard have increased the exchange programs in the last decades, trying to differentiate themselves through the development of very selective double degrees, such as a double-degree in Law between the Harvard Law School and the University of Cambridge, or the double-degree between Harvard Kennedy School and Wharton School, or even those with New York University and with the Graduate Institute Geneva, among others. In the case, h is measured simply by counting the number of partners, then our model is not able to distinguish between these selective connections (*i.e.* double-degrees) and regular exchanges. The lesson is that by measuring the number of connections, h should also measure the *intensity* of the connections in order to better describe the optimal network policy. The expected increase in the number of connections for a given university may actually result in deeper relations and not necessarily more partners, reflecting the idea of more intense relationships as just described.

Intensity can be achieved in one of two ways: either by having privileged (stronger) relations with one partner, or adding more relationships with already existing partners. In the latter case, h would increase without involving other universities. This is a very powerful idea, which enables us to understand the reason why universities decide to create double degrees and enter more strategic alliances and partnerships, instead of expanding the exchange programs with new universities. In fact, the model tells us that connections are important for universities. However, the quality of connections is also as important if not more, from the signalling perspective; in particular, it is more important for high reputation universities than for low reputation ones, since reputation is supposed to be correlated with q. This effect is modelled through the presence of the survival function $\bar{F}_1(s; q_s, l_s)$ in the denominator of the utility. Therefore, high reputation universities have (a) a small, selected number of university partners and (b) numerous (intense) connections with their few partners. They prefer to raise h by increasing the intensity of existing connections instead of connecting with new partners, thus avoiding a decrease of the quality of their connections and of the exclusivity within their network. Thus, the model is also able to explain this fact.

Regarding the quality of connections, the result suggests that the absolute value of quality of connections (q) increases on average over time, while the relative value ($\bar{F}_1(s;q_s,l_s)$) is constant over time. This means that the relative quality of connections that a university has in 1980 is the same as the relative quality of connections it has today. The empirical validity of these results depends on how the quality of connections is defined. The simplest and most direct definition of quality of connections is the arithmetic average of the position of the university partners in comparable rankings. Roughly speaking, a university who had a good quality of connections in the past is likely to keep a good quality of connections today, while a university who had medium quality of connections in the past is likely to have medium quality connections today too no matter how much they have improved - since on average all players have improved. The reasoning behind this behaviour is that each university wants to increase the quality of its own connections, relating with high reputation universities and avoiding bad universities. Thus, bad universities can only enlarge their network with similar bad partners, creating a separate steady state. Overall, who was bad remains bad, who was average tends to remain average and who was good remains good and the relative quality of the connections does not change over time. This is true in the general case and the model is able to explain it.

Furthermore, the fact that the absolute value of the quality of connections (q) is increasing over time has its empirical validity in the comparison with the quality of an average university today and one of a century ago. Considering just the technological facilities available today in almost all the universities it is possible to understand the exponential nature of the growth rate in the absolute value of quality. Another proxy for q can be given by the investments in research of a given university's partners. According to endogenous growth theory, the growth rate of the technological innovation in the model by Romer (1990) or the growth rate of the size of R&D sector in the Jones (1995a,b) critique, which is linked to the increase in investments in R&D due to the arbitrage condition in the model, result to be positive and constant over time in order to explain a positive growth rate of the economy. Thus, these models provide a theoretical and empirical base for the constant growth rate of q.

In addition, there is an ulterior interesting aspect of the result: the impact of uncertainty, as measured by the diffusion term σ in the dynamics of q and h. When this parameter is high, there is high uncertainty on the future values of q and h. For instance, in the deterministic case we have $\sigma = 0$ and the future value of the variables are known for sure. From the assumptions on the parameters, it follows that an increase in σ implies a decrease in γ in order to satisfy the HJB equation in (15). Hence, the growth rate of the quality and the number of connections depends on the uncertainty level of their future value.

This relation reflects the fact that moving in the state space for both variables q and h at each time has an increasing cost, meaning that establishing new connections and improving old ones is increasingly cumbersome. Therefore, in the presence of high uncertainty universities prefer to limit the expansion and improvement of their connections in order to avoid excessive and unpredictable costs that can take place in the future. For this reason, when the uncertainty is high, the optimal drift and the optimal growth rate for both state variables result to be low.

The solution obtained in the model can also be applied with satisfactory results to the empirical case of firms and workers.

Regarding alternative modelling procedures, let us share some of our developments. First, despite the implementation of the model with two different growth rates for q and h, we have decided not to present the model in that form, since the constraint on the integration of the Bellman function would have made them directly proportional to each other. Thus, we would have had two positive and different growth rates for q and h, but at the same time they would have not led to any enhancement of the model or its predictions. More interestingly we have considered the possibility of having different growth rates for each individual depending on the initial value of h_0 and q_0. This probably leads to a solution where the growth rates will depend on time, since the lower bound l_s will behave differently for h_s and q_s over time. However, it is possible that this model will not satisfy the Kolmogorov equation since the different growth rates can probably lead the distribution to explode as time goes to infinity.

5 Conclusion

The model developed in this work describes the optimal behaviour of individuals, which in our case were represented either by universities, firms or workers, in regards to their network of connections. The solution consists on a continuous increase over time of the number and of the absolute value of quality of the connections individuals have. Furthermore, given the optimal effort of each individual, the relative quality of the connections (relative to the quality of the connections of the other individuals) of each individual remains constant over time. The empirical validity of the model has been thoroughly described in the previous section, showing that the model is able to capture different aspects of reality in all the elements of its solution. For the university case, it has been described how the model can capture the increase of the number of connections between universities over time, where in our case the connections between universities can take the form of exchange programs, double degrees or partnerships among others. In particular, the model is able to explain the reason for the creation of different types of strategic partnerships and networks between universities. The model also explains that the most exclusive universities (*i.e.* universities with a relatively low number of connections) in the past, are still today the most exclusive ones, despite the increase in the number of connections. This also holds for the less exclusive universities. Furthermore, the model is also able to capture the increase in the quality of universities over time, which appears clear just comparing the facilities provided by an average university today and one century ago. Moreover, the model helps explaining the fact that universities with good quality connections in the past are the same universities that have good quality connections today; the same applies to medium and low quality universities. Thus, the relative quality of connections remains constant over time, which we believe to be true in general. In addition, the model predicts that when there is uncertainty on the future quality and number of connections then universities tend to limit the expansion and improvement of their connections in order to avoid excessive and unpredictable costs that can take place in the future. Nevertheless we suggest further research to improve the simple model developed in this work.

There are some evident limitations of the model. One is the constant nature of the relative quality over time. Under this prediction it is not possible that universities can increase (or decrease) the quality of their connections over time, because all individuals behave similarly. One possible way of overcoming this problem would be to consider a model where both the drift γ and the number h of connections would be a decreasing function of reputation, and the average quality q of connections would increase with reputation. The model would be different and the implications in the results are not clear. The other limitation is that the model is based on a fixed network of universities, not accommodating the impact of newcomers on this very competitive market. Hence, the model is not able to explain real cases of universities founded just 40 years ago, which are now connected with prestigious universities around the world. However, it is possible to say that in the general case the prediction about the stability of the relative quality of connections holds.

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